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MATERIAL STABILITY AND BIFURCATION IN FINITE ELASTICITY

by

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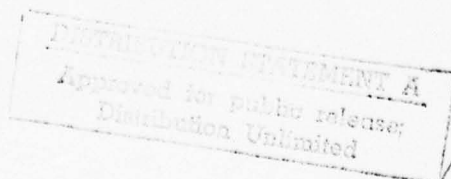
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Material Stability and Bifurcation in Finite Elasticity

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ABSTRACT

The theory of small superposed deformations for isotropic incompressible elastic materials is used (i) to obtain necessary restrictions on the form of the strain-energy function by requiring that the speed of propagation be real for waves that pass through a finitely deformed body of material (i.e., Hadamard stability criterion), and (ii) to determine critical loading conditions for a thick rectangular plate under which bifurcation solutions (i.e., adjacent equilibrium positions) can exist. The possibility of bifurcation under tensile loading, when one pair of faces of a plate are force free, is precluded by further material stability considerations.

1.   Introduction

The main purpose of this paper is to bring together results obtained during the past few years on two basic problems in finite elasticity theory. One of these is the question of what restrictions must be imposed on a constitutive assumption in order to ensure *physically realistic* behavior of the material which it claims to describe. The other problem deals with the description of the conditions under which adjacent equilibrium states can appear as small perturbations on an equilibrium state of finite deformation of a thick plate. Most of the results appear in references [1]-[5].

The setting for both problems is the theory of small deformations superposed on states of pure homogeneous deformation of bodies made of an isotropic, incompressible elastic material [6]. The relevant equations are beiefly derived in §2.

The approach taken for the first problem is to investigate the implications that stem from the Hadamard stability criterion, which requires that the speed of propagation be real for plane waves that pass through a deformed body of material. It was shown by Hayes and Rivlin [7] that this criterion must be imposed in order to ensure that the body can actually exist in the finitely deformed state considered. Accordingly, we adopt the resulting implications as *minimal material stability conditions*.

The equations that describe wave propagation are derived in §3, and further discussed in §4, for a general material. The main result is that certain inequalities [1] {see eqn.(4.1) below} involving derivatives of the strain-energy function are necessary and sufficient for the Hadamard criterion to be satisfied for waves whose directions of propagation lie in a principal plane of strain.

An open question remains as to whether these are also necessary and sufficient when a completely general propagation direction is considered. It is seen, however, that the answer to this is affirmative when two principal extension ratios are equal, or in §5, when the constitutive assumption is specialized [3].

The determination of *critical states* of finite deformation, at which non-trivial adjacent equilibrium states first become possible, has been the subject of numerous investigations over the past twenty years. A partial list of these is contained in [4].

The geometry considered is a rectangular plate, one pair of whose faces are force free and whose other two pairs are loaded by uniformly distributed normal forces when the body is held in its finitely deformed state. The forces acting on one opposite pair of faces are regarded as passive, merely holding the length of the plate fixed in that direction. The forces acting on the other pair are regarded as being the "cause" of the bifurcation, which is assumed to appear as a small plane static deformation parallel to the faces bearing the "passive" forces.

The equations governing such bifurcations are derived in §6, and it is found that further discussion of the problem can be conveniently carried out if two cases that arise naturally are separated. This separation is governed by the particular values assumed by the one relevant material parameter that appears {see (6.15) and (6.18)}. One case [4] is considered in §7 and the other in §8.

When the causal force is a thrust, a single asymptotic formula [5] {see (8.17)}, relating the cross-sectional geometry of a plate to the critical condition for flexural bifurcation, is found to be valid for all materials. This result, for a "thin" cross section, covers the case of classical Euler-type buckling.

When the causal force is tensile, it is found that a bifur-
cation can occur only if the load decreases after attaining a maxi-
mum. However, this behavior rules out the possibility that a plate
of material can be held in stable equilibrium in states of pure
homogeneous deformation just prior to that required for the bifur-
cation, and leads to the imposition of further restrictions on the
strain-energy function beyond those discussed in §4.

A major question that remains open for the bifurcation problem
considered here, particularly for "thick" cross sections, is whether
the body *actually* goes over into a mode shape that is assumed at
the point where the thrust load reaches a critical value. Some
experimental evidence [8] throws doubt on the matter. Attempts to
answer this analytically by employing energetic considerations have,
so far, been unsuccessful.

Recently [9], [10], the bifurcation problem has been exhaus-
tively studied for a plate of incompressible material by employing
an incrementally-linear, time-independent constitutive relationship.
This theory includes small superposed elastic deformations as a
special case. Accordingly, an alternative description of the res-
ults in §§7 and 8 could be gleaned from those in [9] and [10] by
expressing the relevant incremental moduli in terms of the strain-
energy function.

2.   Basic Equations

We let $\underset{\sim}{\xi}$, $\underset{\sim}{X}$ and $\underset{\sim}{\bar{x}}$ denote the vector positions, relative to a fixed rectangular Cartesian frame x, of a typical particle of the body in an undeformed state, in a finitely deformed state, and in a state at time t, respectively, and write

$$\underset{\sim}{\bar{x}} = \underset{\sim}{X} + \underset{\sim}{u} \tag{2.1}$$

where $\underset{\sim}{u}$ is the displacement associated with an additional small superposed deformation. We consider states of pure homogeneous deformation whose principal directions are parallel to the coordinate axes of x and which are completely characterized by the (positive, constant) extension ratios $\lambda_1, \lambda_2, \lambda_3$. Then

$$\underset{\sim}{X} = \underset{\sim}{F}\underset{\sim}{\xi}, \quad \underset{\sim}{F} = \text{diag}(\lambda_1, \lambda_2, \lambda_3) . \tag{2.2}$$

The components of $\underset{\sim}{\xi}$, $\underset{\sim}{X}$, $\underset{\sim}{\bar{x}}$ and $\underset{\sim}{u}$, referred to x, are denoted by ξ_α, X_i, $\bar{x}_i$ and u_i, respectively. In the sequel, all expressions involving $\underset{\sim}{u}$ (and its derivatives) shall be systematically linearized in $\underset{\sim}{u}$.

The Finger strain matrix $\underset{\sim}{\bar{B}}$ is given by

$$\underset{\sim}{\bar{B}} = \underset{\sim}{\bar{F}}\underset{\sim}{\bar{F}}^T = \underset{\sim}{F}^2 + \underset{\sim}{f}\underset{\sim}{F} + \underset{\sim}{F}\underset{\sim}{f}^T , \tag{2.3}$$

where, with (2.1) and (2.2),

$$\underset{\sim}{\bar{F}} = \underset{\sim}{F} + \underset{\sim}{f} , \quad \underset{\sim}{f} = \| u_{i,\alpha} \| . \tag{2.4}$$

(We employ the notation $,_\alpha$ to denote $\partial/\partial\xi_\alpha$, as well as the usual summation convention for repeated lower-case subscripts.) The invariants $\bar{I}_1$ and $\bar{I}_2$ of $\underset{\sim}{\bar{B}}$ are given by

$$\bar{I}_1 = \text{tr} \, \underset{\sim}{\bar{B}} = I_1 + i_1 \, ,$$

$$\bar{I}_2 = \tfrac{1}{2}\{(\text{tr} \, \underset{\sim}{\bar{B}})^2 - \text{tr} \, \underset{\sim}{\bar{B}}^2\} = I_2 + i_2 \, , \qquad (2.5)$$

where

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \, , \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2 \, , \qquad (2.6)$$

$$i_1 = 2\sum_{A=1}^{3} \lambda_A u_{A,A} \, , \quad i_2 = 2\sum_{A=1}^{3} \lambda_A (I_1 - \lambda_A^2) u_{A,A} \, . \qquad (2.7)$$

The incompressibility condition, $\det \underset{\sim}{\bar{F}} = 1$, together with (2.2) and (2.4), yields

$$\lambda_1\lambda_2\lambda_3 = 1 \qquad (2.8)$$

and

$$\sum_{A=1}^{3} \lambda_A^{-1} u_{A,A} = 0 \, . \qquad (2.9)$$

The strain-energy density, W, depends on $\bar{B}$ through $\bar{I}_1$ and $\bar{I}_2$. Letting $\bar{\Pi}_{\alpha i}$ denote the components of the Piola-Kirchhoff stress and $\bar{P}$ a hydrostatic pressure, we have

$$\bar{\Pi}_{\alpha i} + \tfrac{1}{2}\bar{P}\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}\bar{F}_{j\beta}\bar{F}_{k\gamma} = \partial W/\partial\bar{F}_{i\alpha}$$

$$= 2\{(\bar{W}_1 + \bar{I}_1\bar{W}_2)\bar{F}_{i\alpha} - \bar{W}_2\bar{F}_{i\beta}\bar{F}_{j\beta}\bar{F}_{j\alpha}\} \, , \qquad (2.10)$$

where ε_{ijk} is the alternating symbol and

$$\bar{W}_A = \partial W/\partial\bar{I}_A \, , \quad A=1,2 \, . \qquad (2.11)$$

Following the previous notational conventional, we write

$$\bar{\Pi}_{\alpha i} = \Pi_{\alpha i} + \pi_{\alpha i} \ , \quad \bar{P} = P + p \ , \tag{2.12}$$

where $\pi_{\alpha i}$ and p are the stress and hysrostatic pressure, respectively, associated with a superposed deformation. Also, we expand $\bar{W}_A$ in a series about I_1 and I_2 ; thus,

$$\bar{W}_A = W_A + W_{A1} i_1 + W_{A2} i_2 \ , \quad A = 1,2 \tag{2.13}$$

where W_A and W_{AB} denote the first and second derivatives of W with respect to $\bar{I}_A$, evaluated at $\bar{I}_A = I_A$. The substitution from (2.12) and (2.13) into (2.10), with (2.2), (2.4), (2.6) and (2.8), yields

$$\Pi_{AA} = 2\lambda_A \{ W_1 + (I_1 - \lambda_A^2) W_2 \} - P/\lambda_A \ ,$$
$$\Pi_{AB} = 0 \ , \quad A \neq B \ , \quad A,B = 1,2,3 \tag{2.14}$$

and

$$\pi_{AA} = \sum_{B=1}^{3} \{ C_{AB} \delta_{AB} + (\delta_{AB} - 1) D_{AB} + E_{AB} \} u_{B,B} - p/\lambda_A \ ,$$
$$\pi_{AB} = C_{AB} u_{B,A} + D_{AB} u_{A,B} \ , \quad A \neq B \ , \tag{2.15}$$

where δ_{ij} is the Kronecker delta and

$$C_{AB} = 2\{ W_1 + (I_1 - \lambda_A^2 - \lambda_B^2) W_2 \} \ ,$$
$$D_{AB} = (P - 2\lambda_A^2 \lambda_B^2 W_2)/\lambda_A \lambda_B \ , \tag{2.16}$$
$$E_{AB} = 2\lambda_A \lambda_B \{ W_2 + 2 [W_{11} + (2I_1 - \lambda_A^2 - \lambda_B^2) W_{12}$$
$$+ (I_1 - \lambda_A^2)(I_1 - \lambda_B^2) W_{22}] \} \ .$$

(No summation is implied for repeated upper-case subscripts.)

We neglect all body forces. Then, bearing in mind that the
λ's are constants, from (2.1), (2.2), (2.12) and (2.14), the
equations of motion are

$$\pi_{\alpha i, \alpha} = \rho \ddot{u}_i \; , \tag{2.17}$$

together with the requirement that P be constant, where ρ is
the mass density and a superposed dot denotes differentiation with
respect to t . The substitution from (2.15) into (2.17) yields

$$\sum_{B=1}^{3} (C_{AB} u_{A,BB} + E_{AB} u_{B,BA}) - p_{,A}/\lambda_A = \rho \ddot{u}_A \; , \quad A = 1,2,3 \; . \tag{2.18}$$

Let $\underset{\sim}{\nu}$ denote the (outward drawn) unit vector normal to a
surface of the body in the   deformed state. Then the traction
$\underset{\sim}{\bar{T}}$ acting on this surfa   time t , and measured per unit
undeformed area, has components given by

$$\bar{T}_i = \bar{\Pi}_{\alpha i} \nu_\alpha = T_i + t_i \; , \tag{2.19}$$

where

$$T_i = \Pi_{\alpha i} \nu_\alpha \; , \quad t_i = \pi_{\alpha i} \nu_\alpha \; . \tag{2.20}$$

3.   The Propagation of Plane Waves : The Secular Equation

In this section we derive the secular equation that governs the propagation of small amplitude sinusoidal plane waves through a block of material that is otherwise held in equilibrium in the finitely deformed state characterized by $\lambda_1, \lambda_2, \lambda_3$. Letting ω denote the angular frequency, κ the wave number and $\underset{\sim}{n}$ the (real) unit vector in the direction of propagation of such a wave, we employ the usual complex notation and write

$$\underset{\sim}{u} = \underset{\sim}{U} \exp\{\iota(\kappa\underset{\sim}{n}\cdot\underset{\sim}{X}-\omega t)\} , \qquad (3.1)$$

where $\underset{\sim}{U}$ is a constant vector and $\iota^2 = -1$.

The substitution from (3.1) into (2.9), with (2.2), yields

$$\underset{\sim}{n}\cdot\underset{\sim}{U} = n_i U_i = 0 , \qquad (3.2)$$

where we disregard the trivial case $\kappa = 0$.

We express the hydrostatic pressure p associated with the superposed deformation in a form that is consistent with (3.1); thus,

$$p = \iota\kappa q \exp\{\iota(\kappa\underset{\sim}{n}\cdot\underset{\sim}{X}-\omega t)\} , \qquad (3.3)$$

where q is a constant. With the notation

$$\sigma = \kappa^2/(\rho\omega^2) , \qquad (3.4)$$

the substitution from (3.1) and (3.3) into (2.18) yields

$$\sigma(Q_{ij}n_j - qn_i) = U_i , \qquad (3.5)$$

where

$$Q_{ij} = Q_{ijk}U_k , \qquad (3.6)$$

with

$$Q_{ABC} = C_{AB}\lambda_B^2 n_B \delta_{AC} + \lambda_A \lambda_C n_C E_{AC} \delta_{AB} \ . \tag{3.7}$$

To eliminate q from (3.5) we multiply each term of the equation by $\varepsilon_{sri}n_r$ and obtain, with (3.6),

$$\bar{Q}_{st} U_t = 0 \ , \tag{3.8}$$

where

$$\bar{Q}_{st} = \varepsilon_{sri}n_r(\sigma Q_{ijt}n_j - \delta_{it}) \ . \tag{3.9}$$

Now since $n_s \bar{Q}_{st}$ vanishes identically, we note that the three equations in (3.9) are not linearly independent. Taking $s = 1$ and 2, for example, we obtain

$$\bar{Q}_{1t}U_t = 0 \ , \quad \bar{Q}_{2t}U_t = 0 \ , \tag{3.10}$$

which, with (3.2), constitutes a set of three homogeneous equations for $\underset{\sim}{U}$. A non-trivial solution for $\underset{\sim}{U}$ can exist only if

$$\varepsilon_{ijk}\bar{Q}_{1i}\bar{Q}_{2j}n_k = 0 \ . \tag{3.11}$$

The substitution from (3.9) into (3.11), together with some algebraic manipulation, yields the desired secular equation in the form [3]

$$\alpha\sigma^2 - \beta\sigma + 1 = 0 \ , \tag{3.12}$$

where

$$\alpha = \varepsilon_{ijk}n_k n_s n_t(n_1 Q_{2si}Q_{3tj} + n_2 Q_{3si}Q_{1tj} + n_3 Q_{1si}Q_{2tj}) \ ,$$
$$\tag{3.13}$$
$$\beta = n_s(Q_{jsj} - n_j n_k Q_{jsk}) \ ,$$

and where we have used the fact that $\underset{\sim}{n}\cdot\underset{\sim}{n} = 1$. According to

(3.4), the quantity $\rho\sigma$ is the square of the slowness associated

with a wave described by (3.1). We see that (3.12) yields two

positive roots for σ if and only if

$$\alpha > 0 \ , \quad \beta > 0 \ , \quad \text{and} \quad \beta^2 - 4\alpha \geq 0 \ . \tag{3.14}$$

Before turning to a discussion of the implications of (3.14),

we must obtain explicit expressions for α and β in terms of W.

To this end we introduce the notation

$$K_A = W_1 + \lambda_A^2 W_2 \ , \quad M_A = 2(W_{11} + 2\lambda_A^2 W_{12} + \lambda_A^4 W_{22}) \ , \quad A = 1,2,3 \tag{3.15}$$

and also

$$J_A = M_A n_B^2 n_C^2 (\lambda_B^2 - \lambda_C^2)^2 \ , \quad A \neq B \neq C \neq A \ . \tag{3.16}$$

Then employing (2.16) and (3.7) in (3.13), with (2.8), we obtain [3]

$$\alpha = 4(\lambda_1^2 n_1^2 + \#\#)\{(\lambda_1^2 n_1^2 K_2 K_3 + \#\#) + W_1(J_1 + \#\#)\}$$

$$+ 4W_2(\lambda_2^2 \lambda_3^2 J_1 + \#\#)$$

$$+ 16 n_1^2 n_2^2 n_3^2 (\lambda_2^2 - \lambda_3^2)^2 (\lambda_3^2 - \lambda_1^2)^2 (\lambda_1^2 - \lambda_2^2)^2 (W_{11} W_{22} - W_{12}^2) \ , \tag{3.17}$$

$$\beta = 2\{K_1(\lambda_2^2 n_2^2 + \lambda_3^2 n_3^2) + J_1 + \#\#\} \ ,$$

where the symbol $\#\#$ denotes two additional terms which are

obtained from the ones shown by a cyclic permutation of the sub-

scripts 1,2,3 on the λ's, n's, K's, M's and J's. From (3.17),

with (2.8), (3.15) and (3.16), we calculate $\beta^2 - 4\alpha$:

$$= 4W_2^2 \{ [\lambda_1^4 n_1^4 (\lambda_2^2 - \lambda_3^2)^2 + \#\#]$$

$$+ 2[\lambda_2^2 \lambda_3^2 n_2^2 n_3^2 (\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2) + \#\#]\}$$

$$+ 8W_2 \{ J_1 [\lambda_2^2 (\lambda_1^2 - \lambda_3^2)(n_1^2 + n_2^2)$$

$$+ \lambda_3^2 (\lambda_1^2 - \lambda_3^2)(n_1^2 + n_3^2)] + \#\# \}$$

$$+ 4\{ [J_1 + \#\#]^2$$

$$- 16 n_1^2 n_2^2 n_3^2 (\lambda_2^2 - \lambda_3^2)^2 (\lambda_3^2 - \lambda_1^2)^2 (\lambda_1^2 - \lambda_2^2)^2 (W_{11} W_{22} - W_{12}^2) \} . \qquad (3.18)$$

interest to note, in general, that the quantity $\beta^2 - 4\alpha$

etely independent of the value assumed by W_1 .

4.   Stability Restrictions for a General Material

The set of inequalities (3.14), which must hold for all $\underset{\sim}{n}$, is the Hadamard stability criterion. In turn, with (3.17) and (3.18), these inequalities lead to restrictions on the functional form of W , which we shall adopt as *minimal* material stability conditions.

It has been shown [1], [3] that, when the propagation direction $\underset{\sim}{n}$ lies in a principal plane of the underlying pure homogeneous deformation, the necessary and sufficient conditions for (3.14) to be satisfied are

$$K_A > 0 \quad \text{and} \quad (I_1 - \lambda_A^2 - 2\lambda_A^{-1})M_A + K_A > 0 , \quad A = 1,2,3 , \qquad (4.1)$$

where the *K's* and *M's* *are defined by (3.15). A sketch of the* proof of this will be given below.

The first set of conditions (4.1) is known as the Baker-Ericksen inequalities [11]. The second set was apparently first given in [1]. It was shown in [2] that (4.1) can also be derived by considering small static superposed shearing deformations in principal planes and requiring the corresponding incremental shear moduli be positive.

It is not known whether any further restrictions on W , beyond those expressed by (4.1), stem from (3.14) for an arbitrary direction $\underset{\sim}{n}$. However, it is relatively easy to show that (4.1) is *sufficient* to ensure $\beta > 0$ for arbitrary $\underset{\sim}{n}$. We employ $(2.6)_1$, (2.8) and (3.16) in $(3.17)_2$ and write

$$\beta = 2K_1\{(\lambda_2 n_2^2 - \lambda_3 n_3^2)^2 + n_1^2(\lambda_2^2 n_2^2 + \lambda_3^2 n_3^2)\} + \#\# + 2b \ ,$$

$$b = (\lambda_2 + \lambda_3)^2 n_2^2 n_3^2 \{(I_1 - \lambda_1^2 - 2\lambda_1^{-1})M_1 + K_1\} + \#\# \ . \qquad (4.2)$$

From (4.1) we see that $b \geq 0$ and, whence, $\beta > 0$.

In the general case when all three λ's are distinct, we note that, at most, only two of the three Baker-Ericksen inequalities are independent. For, with the ordering $\lambda_1 > \lambda_2 > \lambda_3$, say, from (3.15)$_1$ we have that $K_1 > 0$ and $K_3 > 0$ implies $K_2 > 0$. If two of the λ's are equal, say $\lambda_1 = \lambda_2 \neq \lambda_3$, then, again, at most two of the Baker-Ericksen inequalities are independent while the second set in (4.1) reduces to $K_3 > 0$ and just one other inequality. Thus, for these special types of deformations, (4.1) yields, at most, three independent restrictions on the form of W . Some additional comments pertaining to this case are made at the conclusion of this section.

For propagation in a principal direction, say $\underset{\sim}{n} = (1,0,0)$, (3.17) yields

$$\alpha = 4\lambda_1^4 K_2 K_3 \ , \quad \beta = 2\lambda_1^2(K_2 + K_3) \ , \qquad (4.3)$$

and it is clear that $\beta^2 - 4\alpha \geq 0$. From expressions for α and β analogous to these for propagation along the other two principal directions, we can conclude that $\alpha > 0$ and $\beta > 0$ if and only if $K_A > 0$. Essentially the same results can be read off directly from results obtained much earlier by Ericksen [12].

We turn now to the case of propagation in an arbitrary direction in a principal plane. For definiteness, suppose that $\underset{\sim}{n} = (n_1, n_2, 0)$. Then from (3.17) we obtain

$$\alpha = 4\gamma_1 \gamma_2 \ , \quad \beta = 2(\gamma_1 + \gamma_2) \ , \qquad (4.4)$$

where

$$\gamma_1 = \lambda_1^2 n_1^2 K_2 + \lambda_2^2 n_2^2 K_1 \ ,$$

$$\gamma_2 = K_3 (\lambda_1 n^2 - \lambda_2 n_2^2)^2 \tag{4.5}$$

$$+ \ n_1^2 n_2^2 (\lambda_1 + \lambda_2)^2 \{K_3 + (\lambda_1 - \lambda_2)^2 M_3\} \ .$$

It is clear from (4.4) that $\beta^2 - 4\alpha \geq 0$. With the validity of
$(4.1)_1$ already established, it easily follows [1], [2] from (4.4)
and (4.5) that $\alpha > 0$ if and only if

$$(\lambda_1 - \lambda_2)^2 M_3 + K_3 > 0 \ , \tag{4.6}$$

which, then, is also sufficient for $\beta > 0$, as noted previously.
The form of the conditions $(4.1)_2$ follows from (4.6), with $(2.6)_1$,
by considering analogous results for propagation in the other two
principal planes.

If the underlying deformation is a uniform two-dimensional
extension or compression, we have a situation where two of the
λ's are equal. For example, say

$$\lambda_1 = \lambda_2 = \lambda \ , \quad \lambda_3 = 1/\lambda^2 \ . \tag{4.7}$$

Then, with (3.15), the inequalities (4.1) become

$$K > 0 \ , \quad K_3 > 0 \ , \quad (\lambda^3 - 1)^2 M + \lambda^4 K > 0 \ , \tag{4.8}$$

where

$$K = K_1 = K_2 = W_1 + \lambda^2 W_2 \ , \quad K_3 = W_1 + \lambda^{-4} W_2 \ ,$$

$$M = M_1 = M_2 = 2(W_{11} + 2\lambda^2 W_{12} + \lambda^4 W_{22}) \ . \tag{4.9}$$

The derivatives of W that appear in (4.9) are evaluated for

$$I_1 = 2\lambda^2 + 1/\lambda^4 \ , \quad I_2 = \lambda^4 + 2/\lambda^2 \ , \quad\quad (4.10)$$

which is a point on the curve bounding the domain in the $\bar{I}_1\bar{I}_2$-plane
where W is defined. (See [13], for example).

Now, when two principal extension ratios are equal, we note
that *any* direction $\underset{\sim}{n}$ is parallel to a principal plane, and it
follows that the satisfaction of (4.8) is necessary and sufficient
that (3.14) be satisfied for *arbitrary* directions of propagation
[3].

5.   Results for Some Special Materials

A stumbling block in the way of obtaining the full implications of (3.14) for arbitrary $\underset{\sim}{n}$ is the presence of the term $(W_{11}W_{22}-W_{12}^2)$ in the expression for α . Three special cases where this term is absent have been studied [3]. These are the Mooney-Rivlin material (see, also, [12]) and materials which depend either on $\bar{I}_1$ or on $\bar{I}_2$. For these it is found that the basic restrictions implied by (4.1) are necessary and sufficient for (3.14) to be met for arbitrary $\underset{\sim}{n}$. Only the case where W depends on $\bar{I}_1$ shall be discussed here.

We write $W = f(\bar{I}_1)$ so that, from (3.15) ,

$$K_A = f' \ , \quad M_A = 2f'' \ , \quad A = 1,2,3 \tag{5.1}$$

and, with $(2.6)_1$, the conditions (4.1) become

$$f' > 0 \ , \ 2(\lambda_A - \lambda_B)^2 f'' + f' > 0 \ , \quad AB = 12,23,31 \ . \tag{5.2}$$

The derivatives of f are evaluated at $\bar{I}_1 = I_1$. From (3.17), (3.18), with (5.1), we obtain

$$\alpha = 4(\lambda_1^2 n_1^2 + \#\#)f'\{(\lambda_1^2 n_1^2 + \#\#)f'$$
$$+ 2f''[(\lambda_2^2 - \lambda_3^2)^2 n_2^2 n_3^2 + \#\#]\} \ ,$$
$$\beta = 4\{(\lambda_1^2 n_1^2 + \#\#)f' + f''[(\lambda_2^2 - \lambda_3^2)^2 n_2^2 n_3^2 + \#\#]\} \ , \tag{5.3}$$
$$\beta^2 - 4\alpha = 16\{(\lambda_2^2 - \lambda_3^2)^2 n_2^2 n_3^2 + \#\#\}^2 (f'')^2 \ .$$

It is clear that $\beta^2 - 4\alpha \geq 0$, for arbitrary $\underset{\sim}{n}$, regardless of the form of f . Recalling that $K_A > 0$ is necessary and sufficient

that the speed of waves propagating in a principle direction be real {cf.(4.3)} , it follows from $(5.1)_1$ that $(5.2)_1$ must be satisfied.

Now if $f'' \geq 0$, we see that $(5.2)_2$ is satisfied and also $\alpha > 0$. Therefore, we need consider only such f where $f'' < 0$. Recalling the result cited at the conclusion of §4, we may here assume that all the λ's are distinct, and, without loss of generality, take $\lambda_1 > \lambda_2 > \lambda_3$. Then $(5.2)_2$ yields just one independent condition which can be expressed in the form

$$f'' = -\tfrac{1}{2}(f'-\epsilon)/(\lambda_1-\lambda_3)^2 , \quad \text{with} \quad \epsilon > 0 . \tag{5.4}$$

The substitution from (5.4) into $(5.3)_1$ yields

$$\alpha = 4(\lambda_1^2 n_1^2 + \#\#)f'\{Ff' + \epsilon[(\lambda_2^2-\lambda_3^2)^2 n_2^2 n_3^2 + \#\#]\}/(\lambda_1-\lambda_3)^2 , \tag{5.5}$$

where

$$F = (\lambda_1-\lambda_3)^2(\lambda_1^2 n_1^2 + \#\#) - \{(\lambda_2^2-\lambda_3^2)^2 n_2^2 n_3^2 + \#\#\} . \tag{5.6}$$

According to (5.5) and (5.6) we have $\alpha > 0$ if and only if $\epsilon > 0$ *provided* $F \geq 0$ for all $\underset{\sim}{n}$ and $F = 0$ for some $\underset{\sim}{n}$. That these latter conditions are indeed met can be seen by expressing F as a function of n_1^2 and n_3^2 (recall $n_2^2 = 1-n_1^2-n_3^2$) and then looking for a minumum of F over the interior of the triangular domain in the $n_1^2 n_3^2$-plane bounded by the lines $n_1^2 = 0$, $n_3^2 = 0$, $n_1^2 + n_3^2 = 1$. The relevant conditions are $\partial F/\partial n_1^2 = 0$ and $\partial F/\partial n_3^2 = 0$, which lead to the equations [3]

$$(\lambda_1^2-\lambda_2^2)n_1^2 + (\lambda_3^2-\lambda_2^2)n_3^2 = \{\lambda_1^2-\lambda_2^2-(\lambda_1-\lambda_3)^2\}/2 \ ,$$

$$\tag{5.7}$$

$$(\lambda_1^2-\lambda_2^2)n_1^2 + (\lambda_3^2-\lambda_2^2)n_3^2 = \{\lambda_2^2-\lambda_3^2-(\lambda_1-\lambda_3)^2\}/2 \ .$$

Since $\lambda_1 > \lambda_3$, these are found to be inconsistent, and, hence, no minimum of F exists. It is also found that F > 0 for $n_1^2 = 0$ and $n_3^2 = 0$, but, for $n_1^2+n_3^2=1$, we have

$$F = (\lambda_1-\lambda_3)^2(\lambda_1 n_1 - \lambda_3 n_3)^3 \ , \tag{5.8}$$

which vanishes when

$$n_1^2 = \lambda_3/(\lambda_1+\lambda_3) \ , \quad n_3^2 = \lambda_1/(\lambda_1+\lambda_3) \ . \tag{5.9}$$

We conclude that F > 0 for all $\underset{\sim}{n}$ except for the four values arising from (5.9), for which F = 0 .

6. Bifurcation of a Thick Plate

We consider a rectangular plate having sides of length $2\ell_1$, $2\ell_2$, $2\ell_3$ and which occupies the region

$$-\ell_A \leq \xi_A \leq \ell_A , \quad A = 1,2,3 \tag{6.1}$$

in its undeformed state. The plate is held in equilibrium in a state characterized by (2.2) by means of normal forces uniformly distributed over the faces perpendicular to the x_1 and x_3 directions. No forces are applied to the faces initially normal to the x_2-direction. Here we investigate the possibility that a neighboring equilibrium state can exist in the form of a small superposed displacement in the $x_1 x_2$-plane under the conditions that no further deformation occurs in the x_3-direction, that no tangential tractions are applied to the load-carrying faces, and that these faces remain parallel to their original directions. If such bifurcation is possible, we say that the underlying pure homogeneous deformation is a *critical state*.

With λ_3 fixed, we note from (2.6) and (2.8) that

$$I_1 = \lambda_3^2 + (\lambda+\lambda^{-1})\lambda_3^{-1}, \quad I_2 = \lambda_3(\lambda+\lambda^{-1}) + \lambda_3^{-2} \tag{6.2}$$

where

$$\lambda = \lambda_2/\lambda_1 . \tag{6.3}$$

Also, from (2.14), (2.20) and the fact that the faces $\xi_2 = \pm \ell_2$ are force free, we obtain

$$P = 2\lambda_2^2\{W_1 + (\lambda_1^2+\lambda_3^2)W_2\} , \tag{6.4}$$

and

$$\Pi_{11} = 2(1-\lambda^2)(W_1 + \lambda_3^2 W_2)/(\lambda\lambda_3)^{\frac{1}{2}} . \qquad (6.5)$$

From $(4.1)_1$, with $A = 3$, we see that the force applied to the face $\xi_1 = \ell_1$ is a thrust or tension accordingly as $\lambda > 1$ or $\lambda < 1$, respectively.

To describe neighboring equilibrium states we take $\underset{\sim}{u}$ and p in the form

$$u_1 = u_1(\xi_1,\xi_2), \ u_2 = u_2(\xi_1,\xi_2), \ u_3 = 0, \ p = p(\xi_1,\xi_2) . \qquad (6.6)$$

Then from (2.18) and (2.9) we obtain

$$(C_{11} + E_{11})u_{1,11} + C_{12}u_{1,22} + E_{12}u_{2,21} - p_{,1}/\lambda_1 = 0 ,$$

$$C_{21}u_{2,11} + (C_{22} + E_{22})u_{2,22} + E_{21}u_{1,12} - p_{,2}/\lambda_2 = 0 , \qquad (6.7)$$

$$\lambda u_{1,1} + u_{2,2} = 0 .$$

The condition that the faces $\xi_2 = \pm \ell_2$ remain force free is, from $(2.20)_2$, $\pi_{2i} = 0$. Then with (6.6) and (2.15) we obtain

$$\left.\begin{array}{l} \pi_{21} = C_{21}u_{1,2} + D_{21}u_{2,1} = 0 \\[2mm] \pi_{22} = (C_{22} + E_{22})u_{2,2} + E_{21}u_{1,1} - p/\lambda_2 = 0 \end{array}\right\} \ \xi_2 = \pm\ell_2 . \qquad (6.8)$$

The faces $\xi_1 = \pm \ell_1$ remain parallel to the x_2x_3-plane and are free of tangential tractions if $u_{1,2} = u_{1,3} = 0$ and $\pi_{12} = \pi_{13} = 0$. With (6.6) and (2.15), these conditions are seen to be satisfied if

$$u_{1,2} = u_{2,1} = 0 , \quad \xi_1 = \pm \ell_1 \qquad (6.9)$$

We employ (2.16), (6.3), (6.4) and find that solutions of (6.7), (6.8), (6.9) can be obtained in the separable form [4],

$$u_1 = \left.\begin{matrix} -\sin\Phi\xi_1 \\ \cos\Psi\xi_1 \end{matrix}\right\} U_1(\xi_2), \quad u_2 = \left.\begin{matrix} \cos\Phi\xi_1 \\ \sin\Psi\xi_1 \end{matrix}\right\} U(\xi_2), \quad p = \left.\begin{matrix} \cos\Phi\xi_1 \\ \sin\Psi\xi_1 \end{matrix}\right\} Q(\xi_2) ,$$

(6.10)

where

$$\Phi = n\pi/\ell_1, \quad \Psi = (n-\tfrac{1}{2})\pi/\ell_1, \quad n = 1,2,3,\dots \qquad (6.11)$$

and

$$U_1 = \frac{1}{\lambda\Omega} U', \quad Q = \frac{\lambda_1}{\lambda\Omega^2} \{C_{12}U'' - \Omega^2(C_{11} + E_{11} - \lambda E_{12})U'\} , \qquad (6.12)$$

provided that U satisfies the differential equation

$$U^{(iv)} - \{\lambda^2 + 1 + A(\lambda-1)^2\}\Omega^2 U'' + \lambda^2\Omega^4 U = 0 \qquad (6.13)$$

and the boundary conditions

$$\left.\begin{matrix} U'' + \lambda^2\Omega^2 U = 0 \\ U''' + \Omega^2\{2\lambda^2 + 1 + A(\lambda-1)^2\}U' = 0 \end{matrix}\right\} \quad \xi_2 = \pm \ell_2 \qquad (6.14)$$

The upper set of expressions in (6.10) represents a symmetric mode of deformation with respect to the $x_2 x_3$-plane and the lower set an anti-symmetric mode with respect to this plane. Also, the quantity Ω denotes either Φ or Ψ , depending on which set is considered, and A is defined by

$$A = \frac{2}{\lambda\lambda_3} (\lambda+1)^2 (W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22})/(W_1 + \lambda_3^2 W_2) . \qquad (6.15)$$

It appears that Wesolowski [14] was the first to exploit the

fact that the governing differential equation and boundary conditions for a bifurcation problem of this type depend on the strain-energy function only through a single material parameter, such as A. We remark here that, for any specified value of λ_3, A is a function of λ only {cf.(6.2)} .

With (3.15) we see that A can be expressed as

$A = (\lambda+1)^2 M_3/(\lambda\lambda_3 K_3)$ and, from (4.1) (with the subscript $A=3$) , together with (2.6), (2.8) and (6.3), that the Hadamard criterion can be satisfied only if $(\lambda-1)^2 M_3/(\lambda\lambda_3 K_3) > -1$. Whence, A must satisfy $(\lambda-1)^2 A > -(\lambda+1)^2$, or simply

$$A > -(\lambda+1)^2/(\lambda-1)^2 = A_0(\lambda) , \quad \text{say} , \qquad (6.16)$$

in order that the underlying deformation meet the minimal stability requirement. Of course, no restriction arises from (6.16) if $\lambda=1$.

Upon assuming a solution of (6.13) of the form $U = \exp(\Omega\Gamma\xi_2)$, we obtain a biquadratic equation for Γ which has four roots of the form $\pm (R\pm T)$ where

$$R = \tfrac{1}{2}\{(\lambda+1)^2 + A(\lambda-1)^2\}^{\tfrac{1}{2}} , \quad T = \tfrac{1}{2}\{(A+1)(\lambda-1)^2\}^{\tfrac{1}{2}} . \qquad (6.17)$$

According to (6.16), R is real for all possible values of A and

$$
\begin{aligned}
&T \text{ is real if } A \geq -1 , \\
&T \text{ is imaginary if } -1 > A > -(\lambda+1)^2/(\lambda-1)^2 .
\end{aligned}
\qquad (6.18)
$$

It is convenient to treat these two cases separately. The first is discussed in §7 and the other in §8.

7. Bifurcation When A > - 1

When $(6.18)_1$ applies, we let

$$\Gamma_1 = R + T \ , \quad \Gamma_2 = R - T \tag{7.1}$$

and write the general solution of (6.13) in the form [4]

$$U = L_1 \cosh\Omega\Gamma_1\xi_2 + L_2\sinh\Omega\Gamma_1\xi_2$$
$$+ N_1\cosh\Omega\Gamma_2\xi_2 + N_2\sinh\Omega\Gamma_2\xi_2 \ , \tag{7.2}$$

where L_1, L_2, N_1, N_2 are constants. The substitution into (6.14) yields four equations for the determination of these constants, which, with the notation

$$\eta = \Omega\ell_2 \ , \tag{7.3}$$

can be written as

$$(\Gamma_1^2+\lambda^2)L_1\cosh\Gamma_1\eta + (\Gamma_2^2+\lambda^2)N_1\cosh\Gamma_2\eta = 0 \ ,$$

$$\Gamma_1(\Gamma_2^2+\lambda^2)L_1\sinh\Gamma_1\eta + \Gamma_2(\Gamma_1^2+\lambda^2)N_1\sinh\Gamma_2\eta = 0 \ ,$$

$$(\Gamma_1^2+\lambda^2)L_2\sinh\Gamma_1\eta + (\Gamma_2^2+\lambda^2)N_2\sinh\Gamma_2\eta = 0 \ , \tag{7.4}$$

$$\Gamma_1(\Gamma_2^2+\lambda^2)L_2\cosh\Gamma_1\eta + \Gamma_2(\Gamma_1^2+\lambda^2)N_2\cosh\Gamma_2\eta = 0 \ .$$

In order that non-trivial solutions of (7.4) exist, we must have either

$$L_2 = N_2 = 0 \quad \text{and} \quad \frac{\tanh\Gamma_2\eta}{\tanh\Gamma_1\eta} = \frac{\Gamma_1}{\Gamma_2}\left(\frac{\Gamma_2^2+\lambda^2}{\Gamma_1^2+\lambda^2}\right)^2 \ , \tag{7.5}$$

or

$$L_1 = N_1 = 0 \quad \text{and} \quad \frac{\tanh\Gamma_1\eta}{\tanh\Gamma_2\eta} = \frac{\Gamma_1}{\Gamma_2}\left(\frac{\Gamma_2^2+\lambda^2}{\Gamma_1^2+\lambda^2}\right)^2 \ . \tag{7.6}$$

If (7.5) holds, we can obtain non-trivial values of L_1 and N_1 from (7.4) and, accordingly, from (6.10), (6.12) and (7.2) we see that the variation of u_1 is odd and u_2 is even with respect to ξ_2 . This gives rise to an anti-symmetric shape with respect to the $x_1 x_3$-plane, which we call a *flexural mode*. Conversely, the shape associated with the satisfaction of (7.6) is symmetric with respect to the $x_1 x_3$-plane and we call that a *barreling mode*.

With the convention that

$$\nu = \begin{cases} + 1 \text{ for a flexural mode} \\ - 1 \text{ for a barreling mode ,} \end{cases} \tag{7.7}$$

and with (7.1), it is possible to write both bifurcation conditions (7.5)$_2$ and (7.6)$_2$ in the form [4]

$$\frac{\sinh(2R\eta)}{\sinh(2T\eta)} = \nu \; \frac{R}{T} \; \frac{\{4T^2 + \lambda(\lambda+1)^2\}}{\{4R^2 - \lambda(\lambda-1)^2\}} \; . \tag{7.8}$$

From (6.17), and bearing in mind that A is a function of λ , we see that R and T are functions of λ . Then (7.8) can be regarded as an equation for the determination of η as a function of λ . From (6.11) and (7.3),

$$\eta = n\pi \ell_2 / 2\ell_1 , \quad n = 1,2,3,\ldots \tag{7.9}$$

where n denotes the number of half-wavelengths parallel to the x_1-direction in a neighboring mode. Thus, we may also regard (7.8) as an equation for the determination of critical values of λ corresponding to a particular aspect ratio (ℓ_2/ℓ_1) and a given mode shape.

ain results based on (7.8) have been proven analytically

the assumption that A is *constant*. These are summar-

ollows, where $\theta = 4R^2 - \lambda(\lambda-1)^2$:

If $\lambda > 1$ (tensile case) (7.8) admits no real
solutions and, hence, no bifurcation is possible.

For any $\lambda > 1$, one and only one positive value of
η exists for flexure if $\theta > 0$ and none exists if
$\theta \leq 0$.

For any $\lambda > 1$, one and only one positive value of
η exists for barreling if $\theta < 0$ and none exists if
$\theta \geq 0$.

the vanishing of $\{4R^2 - \lambda(\lambda-1)^2\}$ is the condition which

the ranges of critical values of λ for which flexural

ling bifurcations can occur. From $(6.17)_1$, this separa-

ition is defined by

$$A = \lambda - (\lambda+1)^2/(\lambda-1)^2 = \bar{A}(\lambda), \text{ say } . \qquad (7.10)$$

be said about this in the next section. The function $\bar{A}$

plotted in the λA-plane in Fig.3 of [4] for $A > -1$

3) .

rical results from (7.8) can be obtained [4] by assigning

onstant values to A and then solving numerically for η

tion of λ. This procedure yields curves of the form

ematically in Fig.1. Here we have taken three constant

,b,c for A with the assumed ordering $-1 < a < b < c$.

curves represent solutions of (7.8) for flexure and the

rves the solutions for barreling.

ough the condition A = constant is somewhat artificial

of course, for the Mooney-Rivlin material for which $A \equiv 0$),

from curves such as those in Fig.1 it is possible to determine
critical values of λ , for any specified value of η , for
arbitrary variation of A with λ . The procedure is to read
off pairs of values of critical λ and corresponding A for the
value of η specified and then plot these points in the λA-plane.
This yields two distinct curves, one for flexure and one for bar-
reling. On this same graph we plot the actual variation of A
with λ from (6.15) {recall (6.1) and the fact that λ_3 is fixed}.
The intersections of this latter curve with the former give the
critical values of λ for flexure and barreling corresponding to
the value of η for which the original curves were derived. This
procedure is illustrated in Fig.4 of [4].

We take this opportunity to correct equation (7.4) of [4]
which, for the example chosen, should read

$$A = 2(\lambda+1)^2/(\lambda^2+6\lambda+1) . \tag{7.11}$$

This correction alters slightly the numerical values stated in the
final sentence of [4], but does not change in any way the qualita-
tive aspects of the discussion there. Of course, if the actual
variation of A with λ is known, as for example in (7.11), it
could be employed in (7.8) and a numerical solution effected
directly.

8.   Bifurcation when A < - 1

When $(6.18)_2$ applies, we write

$$T = \iota S , \quad S = \tfrac{1}{2}\{-(A+1)(\lambda-1)^2\}^{\tfrac{1}{2}} , \tag{8.1}$$

and express the general solution of (6.13) in the form

$$U = (L_1 \cos\Omega S\xi_2 + L_2 \sin\Omega S\xi_2)\cosh\Omega R\xi_2$$
$$+ (N_1 \sin\Omega S\xi_2 + N_2 \cos\Omega S\xi_2)\sinh\Omega R\xi_2 , \tag{8.2}$$

where L_1, L_2, N_1, N_2 are constants and R is given by $(6.17)_1$.
We introduce the notation

$$q_1 = R^2 - S^2 + \lambda^2; \quad q_2 = 2RS; \quad q_3 = (1-\lambda)R; \quad q_4 = (1-\lambda)S ;$$

$$Y = \cosh R\eta \cos S\eta ; \quad Z = \sinh R\eta \sin S\eta ; \tag{8.3}$$

$$y = \cosh R\eta \sin S\eta ; \quad z = \sinh R\eta \cos S\eta ;$$

where η is given by (7.3), and then substitute from (8.3) into
(6.14). This yields {cf.(7.4)}

$$\left.\begin{array}{l} L_1(q_1 Y - q_2 Z) + N_1(q_1 Z + q_2 Y) = 0 \\ L_1(q_3 z + q_4 y) + N_2(q_3 y - q_4 z) = 0 \end{array}\right\} , \tag{8.4}$$

$$\left.\begin{array}{l} L_2(q_1 y + q_2 z) + N_2(q_1 z - q_2 y) = 0 \\ L_2(q_3 Z - q_4 Y) + N_2(q_3 Y + q_4 Z) = 0 \end{array}\right\} . \tag{8.5}$$

Now (8.4) and (8.5) admit non-trivial solutions for the L's
and N's *only if* either

$$L_2 = N_2 = 0 \quad \text{and} \quad (q_1 Y - q_2 Z)(q_3 y - q_4 z) = (q_1 Z + q_2 Y)(q_3 z + q_4 y) \tag{8.6}$$

or

$$L_1 = N_1 = 0 \quad \text{and} \quad (q_1 y + q_2 z)(q_3 Y + q_4 Z) = (q_1 z - q_2 y)(q_3 Z - q_4 Y) \ , \quad (8.7)$$

unless it should happen that the determinants associated with (8.4) and (8.5) both vanish simultaneously; that is, *unless*

$$(Yz + Zy)(q_1 q_4 + q_2 q_3) = 0 \ . \tag{8.8}$$

The significance of this exceptional case shall become apparent below.

If (8.6) applies then non-trivial values of L_1 and N_1 can be found from (8.4), which, with (6.10), $(6.12)_1$ and (8.2), gives values for u_1 and u_2 whose variations with ξ_2 are, respectively, odd and even. As before, this situation describes flexural modes. The converse behavior is noted when (8.7) applies which, therefore, governs barreling modes.

We let

$$r = \lambda(3-\lambda)(1+\lambda) + (A+1)(\lambda-1)^2 \ ,$$
$$s = \lambda(\lambda+1) + (A+1)(\lambda-1)^2 \tag{8.9}$$

and substitute from (6.17), (8.1) and (8.3) into (8.6) and (8.7) and cast both bifurcation conditions in the form

$$\nu Rs \sin 2S\eta = Sr \sinh 2R\eta \ , \tag{8.10}$$

where ν is given by (7.7). Bearing in mind that R,S,r,s are functions of λ, we have, as with (7.8), that (8.10) serves to define a relationship between η and critical values of λ.

It can readily be shown that (8.8) is satisfied trivially if $\lambda = 1$ or if $\eta = 0$; but, for $\lambda \neq 1$, there are non-trivial solutions for η of the form

$$\eta = k\bar{\eta} \quad (k=1,2,3,\ldots) \ , \quad \bar{\eta} = \pi/2S \ , \tag{8.11}$$

if and only if

$$r = 0 \ . \tag{8.12}$$

The solutions (8.11) also satisfy (8.10) and, with (8.12), are seen to be *independent* of the value assigned to ν . This leads to the possibility of the simultaneous appearance of flexural and barreling modes, a feature which stands in sharp contrast to that when $A > -1$, where these modes are clearly delineated. The condition (8.12), which gives rise to this feature, is seen, with $(8.9)_1$, to yield an expression for A in terms of λ that is identical in form to $\bar{A}$ in (7.10).

Bearing in mind that λ_3 is fixed, we employ (6.2), (6.5), (6.15) and calculate

$$d\Pi_{11}/d\lambda = -(W_1 + \lambda_3^2 W_2)\{3\lambda^2 + 1 + A(\lambda-1)^2\}/(\lambda^3\lambda_3)^{\frac{1}{2}} \ , \tag{8.13}$$

and note that Π_{11} is stationary for any value of λ which satisfies

$$A = -(3\lambda^2+1)/(\lambda-1)^2 = \tilde{A}(\lambda) \ , \quad \text{say} \ . \tag{8.14}$$

From (6.16), (7.10), (8.14) it easily follows that

$$A_0 < \bar{A} < \tilde{A} \quad \text{for} \quad 0 < \lambda < 1$$

$$\tilde{A} < A_0 < \bar{A} \quad \text{for} \quad \lambda > 1 \tag{8.15}$$

Plots of A_0, $\bar{A}$, $\tilde{A}$ in the λA-plane are shown in Fig.2. The region below the curve A_0 represents values of A which violate (6.16) and, thus, is irrelevant.

To proceed with the discussion of (8.10) we again resort to
the somewhat artificial situation where A is assigned some con-
stant value < - 1 . In doing so, from Fig.2, we see that the
possible values of λ must be restricted to lie in the interval
$e \leq \lambda \leq E$ where e and E are the roots of the equation
$A = A_0(\lambda)$. That is, with (6.16),

$$e = -(1-\sqrt{-A})^2/(A+1) \ , \quad E = -(1+\sqrt{-A})^2/(A+1) \ . \qquad (8.16)$$

We first consider the case where Π_{11} is a thrust $(\lambda>1)$ and then
where it is tensile $(\lambda<1)$.

8.1 The Case of Thrust

Here, for a specified value of $A < -1$, we consider values
of λ that lie in the interval $1 \leq \lambda \leq E$. Let $\bar{\lambda}$ denote the
abscissa of the intersection of the horizontal line corresponding
to this value of A with the curve $\bar{A}$ in Fig.2. Then $1 < \bar{\lambda} < E$,
and we note that $\bar{\lambda}$ and E both increase monotonically with in-
creasing A .

The solutions of (8.10) that result by assigning two values
to A are shown schematically in Fig.3, where values of η are
plotted against critical values of λ . The solid curves pertain
to flexural bifurcation and the broken curves to barreling.
The numerical values of the relevant parameters $\bar{\lambda}$, E and $\bar{\eta}$
corresponding to three representative values of A are listed in
Table I. These can be used to gain some idea of the scale
associated with Fig.3. It can be shown that the *minimum* value of
$\bar{\eta}$ is approximately $2\pi/5$.

Table I.   Relevant parameters for three values of
           A   (thrust)

$\underline{A}$	$\underline{\overline{\lambda}}$	$\underline{E}$	$\underline{\overline{\eta}}$
-2.4625	2.6	4.5135	1.6236
-7.0	2.0	2.2153	1.2825
-17.7778	1.6	1.6360	1.3018

The features common to the solutions of (8.10), for any con-
stant $A < -1$,   are summarized as follows.   Each flexural curve
emanates from the point $\lambda = 1$, $\eta = 0$ and increases steadily to
a point $\lambda = \overline{\lambda}$, $\eta = \overline{\eta}$ where it intersects a downward sloping
barreling curve.   This barreling curve decreases steadily until it
intersects a limiting line $\lambda = E$ at a value of η, η_0 ,   say,
where $\overline{\eta} > \eta_0 > 0$.   No barreling bifurcation is possible if
$\eta < \eta_0$.   The flexure and barreling curves intersect at the points
$\{cf.(8.11)\}$ $\lambda = \overline{\lambda}$, $\eta = k\overline{\eta}(k=1,2,3,...)$,   and each curve crosses
the line $\lambda = \overline{\lambda}$ at these points.   If η lies in any of the open
intervals $(\overline{\eta},2\overline{\eta})$, $(3\overline{\eta},4\overline{\eta})$, $(5\overline{\eta},6\overline{\eta})$,   etc., the critical values for
barreling are less than $\overline{\lambda}$,   while those for flexure, to the extent
they exist, are greater than $\overline{\lambda}$.   Conversely, for values of η in
the intervals $(0,\overline{\eta})$, $(2\overline{\eta},3\overline{\eta})$, $(4\overline{\eta},5\overline{\eta})$, etc., the flexure curve lies
to the left of the line $\lambda = \overline{\lambda}$,   and the barreling curve to the
right.   Both the flexure and barreling curves have the line $\lambda = \overline{\lambda}$
as an asymptote as $\eta \to \infty$.   The barreling and flexure curves may
intersect the line $\lambda = E$;   the tendency for such intersections to
occur increases as   A   decreases (algebraically).

Recalling that η is given by (7.9), it appears that certain
peculiarities arise in connection with the interpretation of the

solutions depicted in Fig.3 for values of λ "close" to $\overline{\lambda}$. If taken at face value, these results for large η can be considered as conditions for the appearance of surface modes in a thick plate [10]. However, pending further investigation into the relevance of the critical conditions for such plates, we here eschew giving any direct physical interpretation.

On the other hand, for values of λ near unity, the flexural curves in Figs.1 and 3 show a strong similarity. This region, which gives the bifurcation condition for a *thin* plate, has been studied [5] for the case $A > -1$, governed by (7.8). The result is expressed by the asymptotic series

$$\lambda = 1 + a_1\eta^2 + a_2\eta^4 + a_3\eta^6 + a_4\eta^8 + O(\eta^{10}) , \tag{8.17}$$

where the a's are given by

$$a_1 = 2/3 , \quad a_2 = 16/45 , \quad a_3 = 2(A + 18/7)/27 ,$$

$$a_4 = 2(A_1 + 113/175)/27 , \tag{8.18}$$

with A_1 denoting the value of A when $\lambda = 1$. From (8.10), following the expansion procedure used in [5], we again obtain (8.17), with (8.18). This result shows (i) that the critical value of λ for flexure is independent of both the form of the strain-energy function and the initial extension ratio λ_3 up to the terms of *fourth* order in η and (ii) that this critical value is extremely insensitive to the detailed behavior of the material up to terms of eighth order in η .

Numerical results for arbitrary variation of A with λ {i.e., for an arbitrary material} could be obtained by plotting curves as in Fig.3 for a large number of constant values of A

and then following the procedure described in [4]. Alternatively, the function A could be employed directly in (8.10).

8.2 The Case of Tensile Load

Here, for a specified value of A < -1 , we consider values of λ that lie in the interval e $\leq$ λ < 1 {cf.(8.16)} , and let $\bar{\lambda},\tilde{\lambda}$ denote the abscissae of the intersections of the horizontal line corresponding to this value of A with the curves $\bar{A}, \tilde{A}$ in Fig.2. Then e < $\bar{\lambda}$ < $\tilde{\lambda}$ < 1 , and we note that e, $\bar{\lambda}$, $\tilde{\lambda}$ all decrease monotonically as A increases.

In Fig.4 are shown, schematically, the solutions of (8.10) that result by assigning two values to A . Again, the solid curves pertain to flexure and the broken curves to barreling. Wesolowski [14] restricted attention to barreling bifurcations. To obtain some idea of the scale associated with Fig.4, the numerical values of e, $\bar{\lambda},\tilde{\lambda},\bar{\eta}$, corresponding to three representative values of A are listed in Table II.

The most important feature to note in connection with any solution of the type shown in Fig.4 is that neither the barreling nor the flexural curve has any point lying to the right of the line $\lambda = \tilde{\lambda}$. Thus, there is a range of λ , $\tilde{\lambda}$ < λ < 1 , for which *no* bifurcation can occur.

Table II. Relevant parameters for three values of A (tensile)

A	e	$\bar{\lambda}$	$\tilde{\lambda}$	$\bar{\eta}$
-2.05	0.1776	0.2	0.2425	3.8324
-7.0	0.4514	0.4641	0.5	2.3933
-15.4	0.5938	0.6	0.6246	2.0697

If we assume for the moment a material does exist for which

A is constant and which also meets the minimal material stability requirement $(W_1 + \lambda_3^2 W_2) > 0$, then from (6.5), (8.13) and (8.14) it follows that the tensile force Π_{11} increases from zero to a maximum as λ decreases from unity to $\tilde{\lambda}$, and then decreases as λ continues to decrease. Whence, in concurrence with Wesolowski [14] and Hill and Hutchinson [9], we conclude that if a bifurcation is to occur, it must do so after a load maximum is achieved. However, this behavior can be used to illustrate that a plate of this material having a finite aspect ratio (ℓ_2/ℓ_1) could behave as a perpetual motion machine without suffering a bifurcation.

The extension of this argument to a general material leads to the conclusion that, if the relevant variation of A with λ is plotted in Fig.2, the resulting curve must lie *above* the curve $\tilde{A}$ for $0 < \lambda < 1$. Thus, in general, no bifurcations of the types considered can exist under tensile loading conditions.

We remark that the curve $\tilde{A}$, itself, is independent of the value of the extension ratio λ_3 . Also, the above conclusion is based solely on the implications arising from (6.5), (8.13) and the necessity of the Baker-Ericksen inequalities, and leads to a further restriction on the form of the strain-energy function beyond that implied by $(4.1)_2$. This restriction, viz. $A > \tilde{A}$, can be cast in the form

$$(\lambda_1^2 - \lambda_2^2)^2 M_3 + (\lambda_1^2 + 3\lambda_2^2) K_3 > 0 , \quad (\lambda_1 > \lambda_2) \tag{8.19}$$

upon employing (2.8), (3.15), (6.3), (6.13) and (8.14). In the event that the passive load applied to the faces $\xi_3 = \pm \ell_3$ of the plate is also tensile (i.e., if $\lambda_3 > \lambda_2$), then a similar

line of reasoning, with the roles of λ_1 and λ_3 interchanged, leads to the conclusion that another of the conditions $(4.1)_2$ must be replaced by the stronger restriction

$$(\lambda_3^2-\lambda_2^2)^2 M_1 + (\lambda_3^2+3\lambda_2^2) K_1 > 0 , \quad (\lambda_3>\lambda_2) . \qquad (8.20)$$

Acknowledgement

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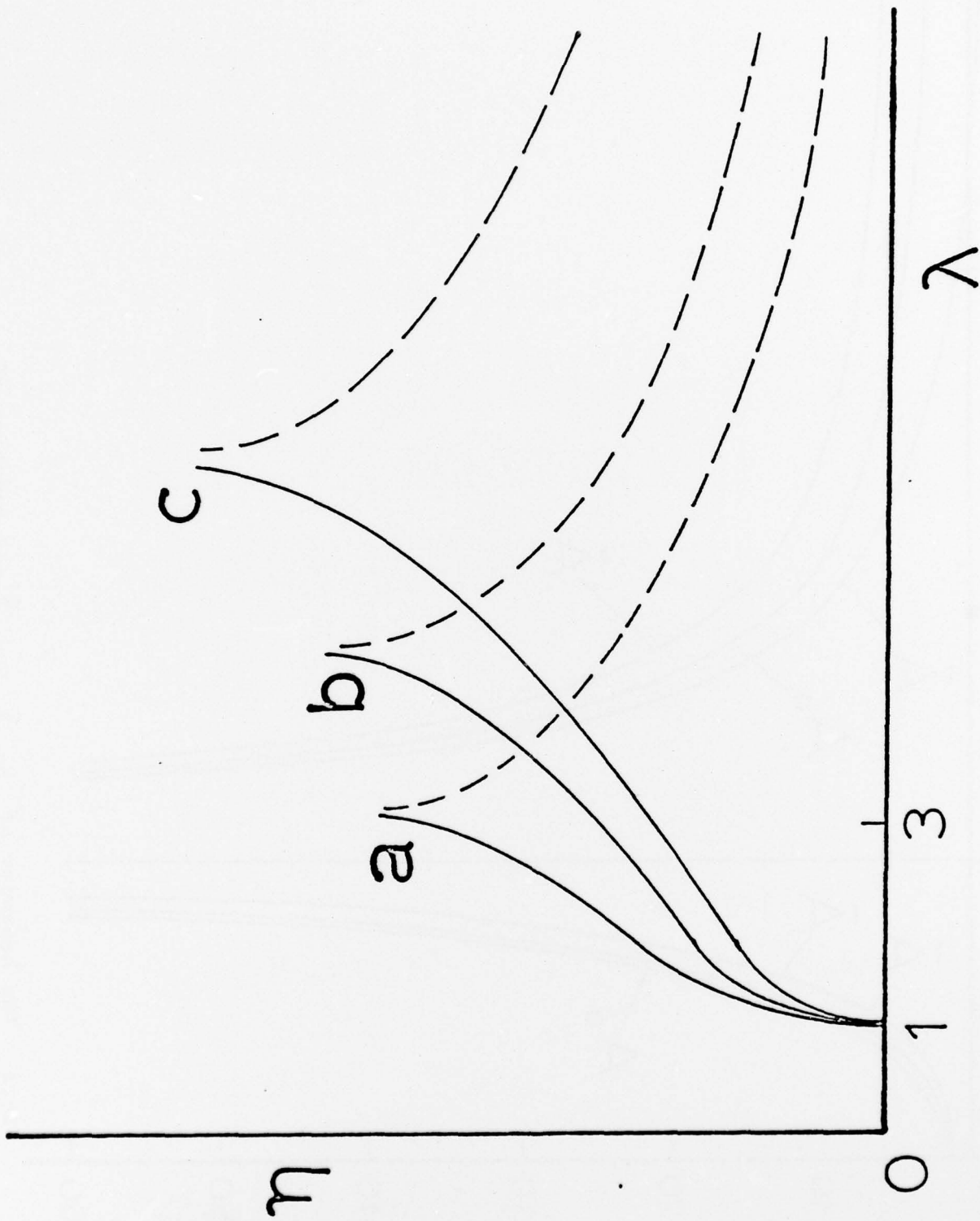[14] Z. Wesolowski, Arch. Mech. Stosowanej, 14, 875 (1962).

Fig.1. Plots of η vs. critical λ for three typical values of $A > -1$ $(-1 < a < b < c)$.

Fig.2. The behavior of A_0, $\bar{A}$ and $\tilde{A}$ as function of λ.
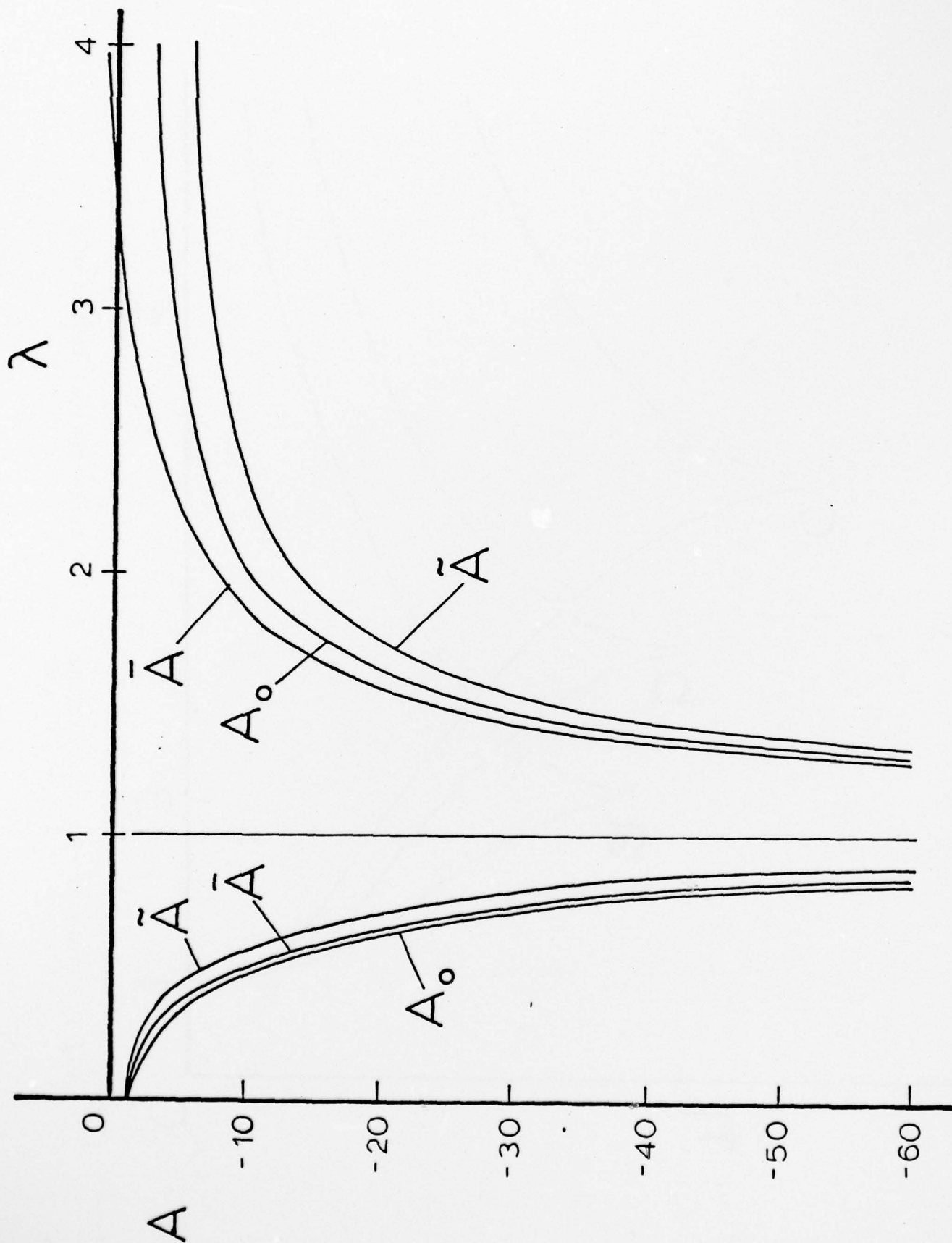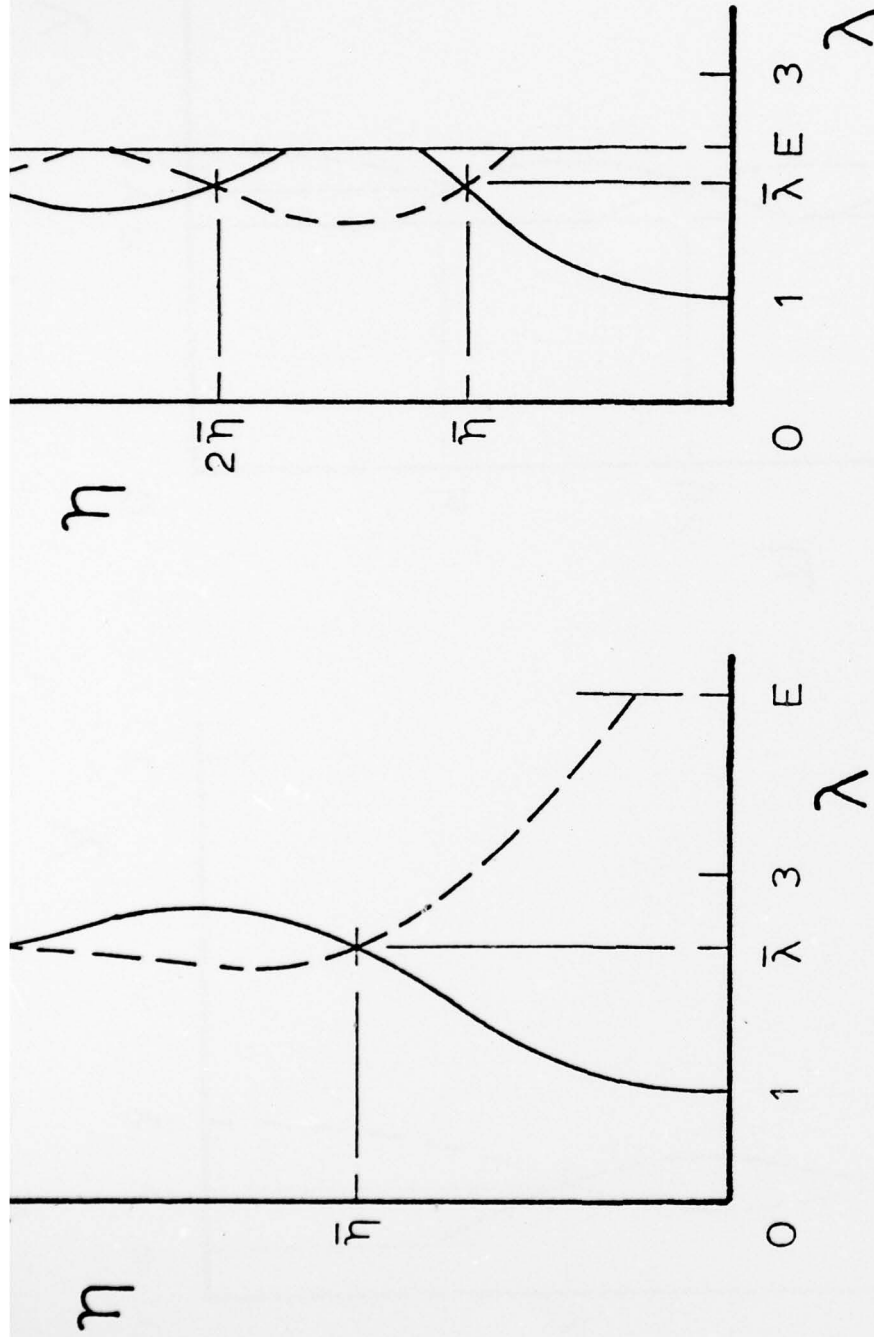
(i) $A = a$ (ii) $A = b$

Fig.3. Plots of η vs. critical λ for two typical values of $A < -1$
under thrust loading.
(i) $A = a$ (ii) $A = b$, with $-1 > a > b$.

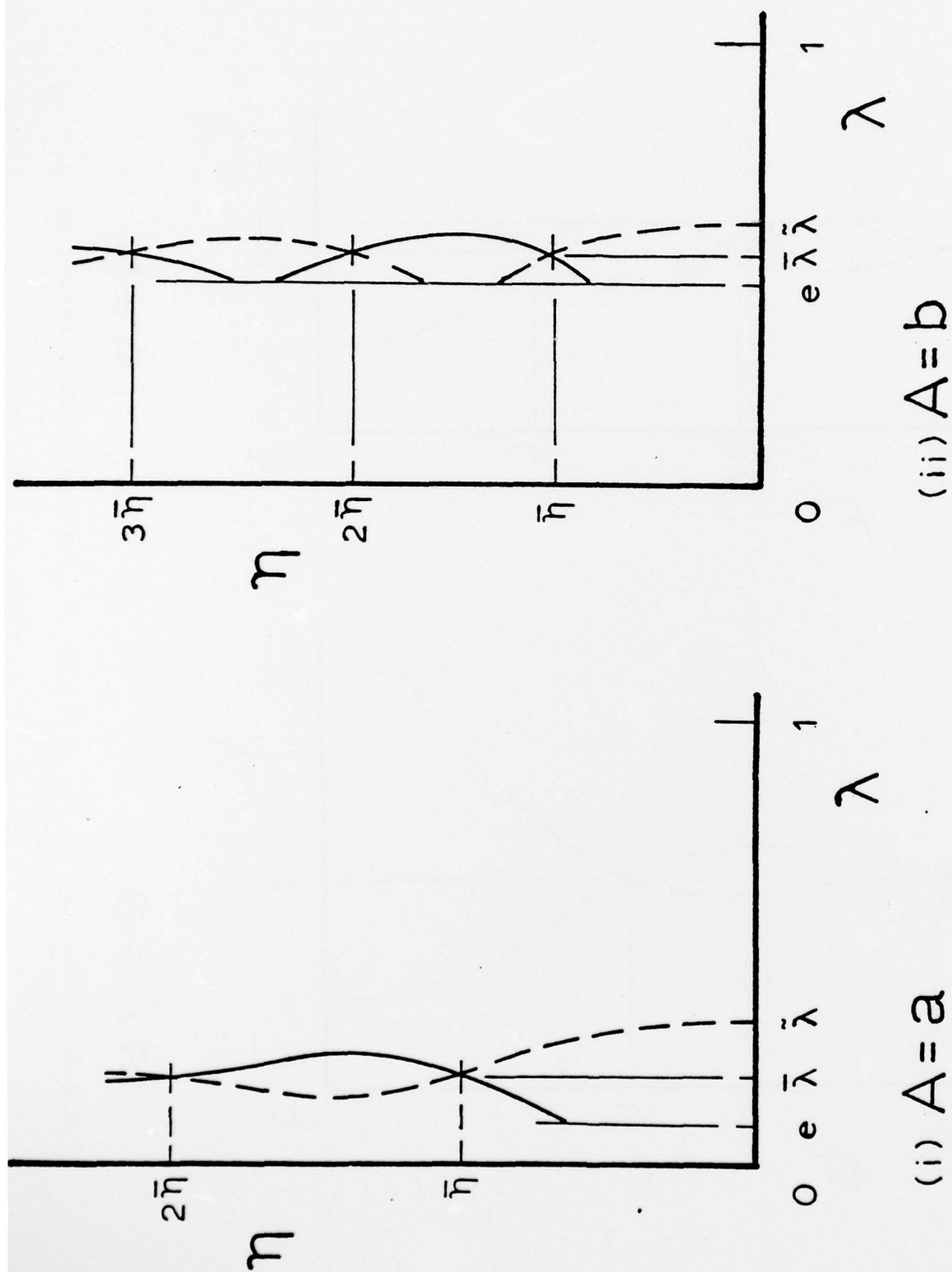Fig. 4. Plots of η vs. critical λ for two typical values of $A < -1$ under tensile loading. (i) $A = a$, (ii) $A = b$, with $-1 > a > b$.

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13. ABSTRACT

The theory of small superposed deformation for isotropic incompressible elastic materials is used (i) to obtain necessary restriction on the form of the strain-energy function by requiring that the speed of propagation be real for waves that pass through a finitely deformed body of material (i.e., Hadamard stability criterion), and (ii) to determine critical loading conditions for a thick rectangular plate under which bifurcation solutions (i.e., adjacent equilibrium positions) can exist. The possibility of bifurcation under tensile loading, when one pair of faces of a plate are force free, is precluded by further material stability considerations.

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